Asymptotic structure of topologically massive gravity in spacelike stretched AdS sector

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Asymptotic structure of topologically massive gravity in spacelike stretched AdS sector

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Abstract: We introduce a natural set of asymptotic conditions in the spacelike stretched AdS sector of topologically massive gravity. The Poisson bracket algebra of the canonical generators is shown to have the form of the semi-direct sum of a $u(1)$ Kac-Moody and a Virasoro algebra, with central charges. Using the Sugawara construction, we prove that the asymptotic symmetry coincides with the conformal symmetry, described by two independent Virasoro algebras with central charges. The result is in complete agreement with the hypothesis made in [6].

Keywords: Classical Theories of Gravity, Conformal and W Symmetry, Space-Time Symmetries

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Contents

1 Introduction

Topologically massive gravity with a cosmological constant Λ, denoted shortly as TMG_Λ, is an extension of three-dimensional general relativity with a cosmological constant (GR_Λ) by a gravitational Chern-Simons term [1]. While GR_Λ is a topological theory, TMG_Λ is a dynamical theory with one propagating mode, the massive graviton. In the AdS sector (with $\Lambda < 0$), TMG_Λ contains a maximally symmetric vacuum solution, known as AdS_3, and the related BTZ black hole [2], with interesting thermodynamic properties. Thus, TMG_Λ seems to be a useful model for exploring dynamical properties of the gravitational dynamics. However, the interpretation of TMG_Λ for generic values of the Chern-Simons coupling constant suffers from serious difficulties: for the usual sign of the gravitational coupling constant, $G > 0$, massive excitations about AdS_3 carry negative energy [1], while $G < 0$ leads to the negative energy of the BTZ black hole [3].

In order to resolve this inconsistency, Li et. al. [3] introduced the so-called chiral version of the theory, defined by a specific relation between the coupling constants, and argued that it might lead to a consistent theory at both classical and quantum level [3, 4]. Here, we follow another idea, related to the fact that TMG_Λ has a rather rich vacuum structure [5, 6]. Since the AdS sector of TMG_Λ around AdS_3 is not consistent, Anninos et al. [6] proposed to choose a new vacuum, the so-called spacelike stretched AdS_3, which could be a stable ground state of the theory [7]. This choice reduces the isometry group $\mathrm{SL}(2, R) \times \mathrm{SL}(2, R)$ of AdS_3 to its four parameter subgroup $\mathrm{U}(1) \times \mathrm{SL}(2, R)$. Exploring thermodynamic properties of the spacelike stretched black hole, Anninos et al. [6] were led to a hypothesis that the corresponding boundary dynamics is described by a *holographically dual two-dimensional conformal field theory* (as in the standard AdS_3 case). Recently, an extension of the above hypothesis to the dS sector was discussed in [8].

As a natural step toward verification of the above hypothesis, Compère et al. [9] (see also [10]) investigated asymptotic symmetries in the spacelike stretched AdS_3 sector. They found a structure isomorphic to the semi-direct sum of the $u(1)$ Kac-Moody algebra and the Virasoro algebra, $u(1)_{KM} \oplus_{sd} V$, with a central extension. The result looks quite natural, but the validity of the hypothesis still remains an open issue.

In this paper, we also examine the correctness of the hypothesis formulated in [6]. Since we are convinced that the asymptotic structure of a dynamical system is most clearly seen in the canonical formalism, our approach is based on Dirac's constraint Hamiltonian formalism, in a form applied recently to TMG_Λ [11]. After formulating a set of natural asymptotic conditions that generalize the usual AdS conditions [12], we find that the asymptotic symmetry of the spacelike stretched AdS sector of TMG_Λ is indeed a two-dimensional *conformal symmetry* with central charges, in complete agreement with the hypothesis of [6].

The paper is organized as follows. In section 2 we give a brief overview of the basic dynamical features of TMG_Λ in the first-order Lagrangian formalism and discuss the form of the spacelike stretched black hole solution. Then, in section 3, we formulate the concept of *asymptotically warped AdS configuration*, derive the asymptotic conditions in the spacelike stretched sector, and find the form of the corresponding asymptotic parameters. The asymptotic commutator algebra is found to be $u(1)_{KM} \oplus_{sd} V$ *without* central charges. In section 4, we study the canonical realization of the asymptotic symmetry by constructing the Poisson bracket algebra of the improved canonical generators. It turns out that it has the form $u(1)_{KM} \oplus_{sd} V$ *with* central charges. This algebra is essentially of the same form as the one found in [9, 10]. In section 5, we derive the main result of this paper: using the Sugawara construction [13] in the $u(1)_{\mathrm{KM}}$ sector, we find that the asymptotic symmetry can be written in the form of *two independent Virasoro algebras with central charges*, the values of which coincide with those conjectured in [6]. Section 6 is devoted to concluding remarks, while appendices contain some technical details.

Our conventions are given by the following rules: the Latin indices refer to the local Lorentz frame, the Greek indices refer to the coordinate frame; the middle alphabet letters $(i, j, k, \ldots; \mu, \nu, \lambda, \ldots)$ run over 0,1,2, the first letters of the Greek alphabet $(\alpha, \beta, \gamma, \ldots)$ run over 1,2; the metric components in the local Lorentz frame are $\eta_{ij} = (+,-,-)$; totally antisymmetric tensor ε^{ijk} and the related tensor density $\varepsilon^{\mu\nu\rho}$ are both normalized as $\varepsilon^{012} = 1$.

2 Spacelike stretched black holes

Topologically massive gravity with a cosmological constant is formulated as a gravitational theory in Riemannian spacetime. Instead of using the standard Riemannian formalism, with an action defined in terms of the metric, with find it more convenient to work in the first-order formalism, with the triad field and spin connection as fundamental dynamical variables. Such an approach can be naturally described in the framework of Poincaré gauge theory [14], where basic gravitational variables are the triad field b^i and the Lorentz connection $A^{ij} = -A^{ji}$ (1-forms), and the corresponding field strengths are the torsion T^i and the curvature R^{ij} (2-forms). Using the notation $A^{ij} =: -\varepsilon^{ij}{}_k\omega^k$ and $R^{ij} =: -\varepsilon^{ij}{}_k R^k$, we have: $T^i = \nabla b^i := db^i + \varepsilon^i{}_{jk}\omega^j b^k$ and $R^i = d\omega^i + \frac{1}{2}\,\varepsilon^i{}_{jk}\omega^j\omega^k$ (the wedge product sign is omitted for simplicity).

The antisymmetry of A^{ij} ensures that the underlying geometric structure corresponds to Riemann-Cartan geometry, in which b^i is an orthonormal coframe, $g := \eta_{ij} b^i \otimes b^j$ is the metric of spacetime, ω^i is the Cartan connection, and T^i, R^i are the torsion and the Cartan curvature, respectively. For $T_i = 0$, this geometry reduces to Riemannian.

2.1 Lagrangian and the field equations

The Lagrangian of TMG_Λ is defined by

$$L = 2ab^i R_i - \frac{\mathit{\Lambda}}{3}\,\varepsilon_{ijk} b^i b^j b^k + a\mu^{-1} L_{\mathrm{CS}}(\omega) + \lambda^i T_i\,, \tag{2.1}$$

where $a = 1/16\pi G$, $L_{\mathrm{CS}}(\omega) = \omega^i d\omega_i + \frac{1}{3}\varepsilon_{ijk}\omega^i\omega^j\omega^k$ is the Chern-Simons Lagrangian for the Lorentz connection, and λ^i (1-form) is the Lagrange multiplier that ensures $T_i = 0$. We assume that $G > 0$, while the values of μ are generic. By construction, TMG_Λ is invariant under the local Poincaré transformations:

$$\begin{aligned}
\delta_0 b^i{}_\mu &= -\varepsilon^i{}_{jk} b^j{}_\mu \theta^k - (\partial_\mu \xi^\rho) b^i{}_\rho - \xi^\rho \partial_\rho b^i{}_\mu\,,\\
\delta_0 \omega^i{}_\mu &= -\nabla_\mu \theta^i - (\partial_\mu \xi^\rho)\omega^i{}_\rho - \xi^\rho \partial_\rho \omega^i{}_\mu\,,\\
\delta_0 \lambda^i{}_\mu &= -\varepsilon^i{}_{jk}\lambda^j{}_\mu \theta^k - (\partial_\mu \xi^\rho)\lambda^i{}_\rho - \xi^\rho \partial_\rho \lambda^i{}_\mu\,.
\end{aligned} \tag{2.2}$$

By varying the action $I = \int L$ with respect to b^i, ω^i and λ^i, one obtains the gravitational field equations. Using the third equation $T_i = 0$, which ensures that ω^i is the Riemannian (Levi-Civita) connection, the first two equations can be written as

$$2aR_i - \mathit{\Lambda}\varepsilon_{ijk} b^j b^k + 2a\mu^{-1} C_i = 0\,, \tag{2.3a}$$

$$\lambda_m = 2a\mu^{-1} L_{mn} b^n\,, \qquad L_{mn} := (Ric)_{mn} - \frac{1}{4}\eta_{mn} R\,. \tag{2.3b}$$

Here, $C_i := \nabla(L_{ik} b^k)$ is the Cotton 2-form, $(Ric)_{mn} = -\varepsilon^{kl}{}_m R_{kln}$ and $R = -\varepsilon^{ijk} R_{ijk}$. The expansion in the basis $\hat\epsilon_k = \frac{1}{2}\varepsilon_{kmn} b^m b^n$, given by $R_i = G^k{}_i\hat\epsilon_k$, $C_i = C^k{}_i\hat\epsilon_k$, yields the standard component form of the first equation:

$$aG_{ij} - \mathit{\Lambda}\eta_{ij} + a\mu^{-1} C_{ij} = 0\,,$$

where G_{ij} is the Einstein tensor, and $C_{ij} = \varepsilon_i{}^{mn}\nabla_m L_{nj}$ the Cotton tensor.

2.2 Construction of spacelike stretched black holes

The spacelike stretched black hole [5, 6] is a particular solution of TMG_Λ with several attractive features: it is a discrete quotient of the spacelike stretched AdS_3 vacuum (A.1), both solutions have the same type of asymptotic behaviour, and the corresponding black hole thermodynamics [15] seems to support the hypothesis made in [6], which "predicts" the existence of an asymptotic conformal symmetry in this sector of TMG_Λ.

Using the results described in appendix A, we are now going to construct the spacelike stretched black hole in our first-order formalism. After introducing a convenient notation,

$$\Lambda = -\frac{a}{\ell^2}\,, \qquad \nu = \frac{\mu\ell}{3}\,,$$

we start from the spacelike stretched AdS_3 solution (A.1), use the coordinate transformations (A.3) and find the form of the spacelike stretched black hole metric in Schwarzschild-like coordinates $x^\mu = (t, r, \varphi)$:

$$ds^2 = N^2dt^2 - B^{-2}dr^2 - K^2(d\varphi + N_\varphi dt)^2\,, \tag{2.4}$$

where

$$\begin{aligned}
N^2 &= \frac{(\nu^2+3)(r-r_+)(r-r_-)}{4K^2}\,, \qquad B^2 = \frac{4N^2K^2}{\ell^2}\,,\\
K^2 &= \frac{r}{4}\left[3(\nu^2-1)r + (\nu^2+3)(r_+ + r_-) - 4\nu\sqrt{r_+r_-(\nu^2+3)}\right]\,,\\
N_\varphi &= \frac{2\nu r - \sqrt{r_+r_-(\nu^2+3)}}{2K^2}\,.
\end{aligned}$$

The metric of the spacelike stretched black hole (2.4) is defined for $\nu^2 > 1$.

Going over to the first-order formalism, we choose the triad field to have the simple diagonal form:

$$b^0 = N dt\,, \qquad b^1 = \frac{\ell}{2NK}dr\,, \qquad b^2 = K(d\varphi + N_\varphi dt)\,. \tag{2.5a}$$

The connection ω^i is determined by the condition of vanishing torsion, $db^i + \varepsilon_{ijk}\omega^j b^k = 0$:

$$\omega^0 = -\beta b^0 - \gamma b^2\,, \qquad \omega^1 = -\beta b^1\,, \qquad \omega^2 = -\alpha b^0 + \beta b^2\,, \tag{2.5b}$$

where

$$\alpha := \frac{2KN'}{\ell}\,, \qquad \beta := \frac{K^2N'_\varphi}{\ell}\,, \qquad \gamma := \frac{2NK'}{\ell}\,.$$

In the coordinate basis, we have:

$$\begin{aligned}
\omega^0 &= -\frac{N\nu}{\ell}dt - \frac{2NKK'}{\ell}d\varphi\,, \qquad\qquad \omega^1 = -\frac{KN'_\varphi}{2N}dr\,,\\
\omega^2 &= -\frac{KN_\varphi\nu}{\ell}dt + \frac{K^3N'_\varphi}{\ell}d\varphi\,.
\end{aligned}$$

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Finally, the solution for λ_m takes the form:

$$\begin{aligned}
\lambda_0 &= 2a\mu^{-1}\left[\left((Ric)_{00}-\frac{3}{2\ell^2}\right)b^0+(Ric)_{02}b^2\right] ,\\
\lambda_1 &= 2a\mu^{-1}\left((Ric)_{11}+\frac{3}{2\ell^2}\right)b^1 ,\\
\lambda_2 &= 2a\mu^{-1}\left[(Ric)_{20}b^0+\left((Ric)_{22}+\frac{3}{2\ell^2}\right)b^2\right] ,
\end{aligned} \tag{2.5c}$$

where the Ricci tensor $(Ric)_{ij}$ is calculated in appendix B. Equations (2.5) define the spacelike stretched black hole in the first-order formalism.

3 Asymptotic conditions

In this section, we use a natural technique, known from earlier studies of the AdS sector [12], to introduce asymptotic conditions in the sector containing the spacelike stretched AdS_3, then we analyze the corresponding restrictions on the gauge parameters and calculate the form of the commutator algebra.

3.1 Spacelike stretched AdS asymptotics

Let us introduce the concept of *warped AdS asymptotic behavior*, based on the following requirements:

(a) asymptotic configurations should include warped black hole geometries;

(b) they should be invariant under the action of $\mathrm{U}(1)\times \mathrm{SL}(2,R)$, the isometry group of warped AdS_3;

(c) asymptotic symmetries should have well-defined canonical generators.

Here, we apply this general concept to the case of spacelike stretched AdS asymptotics.

The requirement (a) means that asymptotic conditions should be chosen so as to include the spacelike stretched black hole configuration, defined by (2.5).

In order to realize the requirement (b), we first consider the spacelike stretched black hole metric (2.4). For large r, this metric can be written in the form $g_{\mu\nu}=\bar g_{\mu\nu}+\tilde G_{\mu\nu}$, where $\bar g_{\mu\nu}$ is the leading-order term (the black hole vacuum, defined by $r_-=r_+=0$),

$$\bar g_{\mu\nu} = -\begin{pmatrix} 1 & 0 & \nu r \\ 0 & \dfrac{\ell^2}{(\nu^2+3)r^2} & 0 \\ \nu r & 0 & \dfrac{3}{4}(\nu^2-1)r^2 \end{pmatrix} ,$$

and $\tilde G_{\mu\nu}$ represents the sub-leading terms. Let us now act on $g_{\mu\nu}$ with all possible isometries of the spacelike warped AdS_3, defined by the four Killing vectors $\xi_K=(\xi_{(2)},\bar\xi_{(1)},\bar\xi_{(2)},\bar\xi_{(0)})$, displayed in appendix A. The result of this procedure has the form

$$\delta_K g_{\mu\nu} = \begin{pmatrix} 0 & \mathcal{O}_2 & \mathcal{O}_0 \\ \mathcal{O}_2 & \mathcal{O}_3 & \mathcal{O}_1 \\ \mathcal{O}_0 & \mathcal{O}_1 & \mathcal{O}_{-1} \end{pmatrix} ,$$

where $\mathcal{O}_n$ is a quantity that tends to zero as $1/r^n$ or faster when $r \to \infty$. In order to have a set of asymptotic configurations which is sufficiently large to include the whole family of the metric configurations $\bar{g}_{\mu\nu} + \delta_K g_{\mu\nu}$, as required by (b), we adopt the following asymptotic form of the metric:

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + G_{\mu\nu}\,, \qquad G_{\mu\nu} = \begin{pmatrix} \mathcal{O}_1 & \mathcal{O}_2 & \mathcal{O}_0 \\ \mathcal{O}_2 & \mathcal{O}_3 & \mathcal{O}_1 \\ \mathcal{O}_0 & \mathcal{O}_1 & \mathcal{O}_{-1} \end{pmatrix}. \tag{3.1}$$

Comparing this result with [10], we find a complete agreement.

To simplify further discussion, we will use the notation $\bar{\phi} := \phi(r_- = r_+ = 0)$ for the leading-order term of any dynamical variable ϕ, which is a natural extension of the notation used for the metric.

Although metric is not a dynamical variable in our first order formalism, its asymptotic form can be used to "derive" asymptotic behaviour of the triad field. Indeed, by combining (2.5a) and (3.1), we are led to adopt the following asymptotic form of the triad field:

$$b^i{}_\mu = \bar{b}^i{}_\mu + B^i{}_\mu\,, \qquad B^i{}_\mu := \begin{pmatrix} \mathcal{O}_1 & \mathcal{O}_2 & \mathcal{O}_2 \\ \mathcal{O}_2 & \mathcal{O}_2 & \mathcal{O}_1 \\ \mathcal{O}_1 & \mathcal{O}_2 & \mathcal{O}_0 \end{pmatrix}. \tag{3.2a}$$

Similarly, we combine (3.2a) with (2.5b) to find

$$\omega^i{}_\mu = \bar{\omega}^i{}_\mu + \Omega^i{}_\mu\,, \qquad \Omega^i{}_\mu := \begin{pmatrix} \mathcal{O}_1 & \mathcal{O}_2 & \mathcal{O}_0 \\ \mathcal{O}_2 & \mathcal{O}_2 & \mathcal{O}_1 \\ \mathcal{O}_1 & \mathcal{O}_2 & \mathcal{O}_0 \end{pmatrix}. \tag{3.2b}$$

Finally, by combining (3.2a) with (2.5c), we obtain:

$$\lambda^i{}_\mu = \bar{\lambda}^i{}_\mu + \Lambda^i{}_\mu\,, \qquad \Lambda^i{}_\mu := \begin{pmatrix} \mathcal{O}_1 & \mathcal{O}_2 & \mathcal{O}_0 \\ \mathcal{O}_2 & \mathcal{O}_2 & \mathcal{O}_1 \\ \mathcal{O}_1 & \mathcal{O}_2 & \mathcal{O}_0 \end{pmatrix}. \tag{3.2c}$$

One should note that the asymptotic conditions are not uniquely determined by the requirements (a) and (b). In the above procedure, we were looking for the most general asymptotic behaviour compatible with (a) and (b), with arbitrary higher-order terms. Later, when we consider the condition (c), certain relations among the higher-order terms will be established (in appendix C).

By construction, the adopted asymptotic conditions are invariant under the action of the isometry group $\mathrm{U}(1) \times \mathrm{SL}(2,R)$ of the spacelike warped AdS_3. Now, we wish to clarify the symmetry structure of the field configurations (3.2).

3.2 Asymptotic parameters

Having chosen the asymptotic conditions (3.2), we are now going to find the subset of gauge transformations (2.2) that leave these conditions invariant. More precisely, acting on the fields (3.2), these restricted (or asymptotic) gauge transformations are, by definition, allowed to change only the (arbitrary) higher-order terms. Consequently, the restricted gauge parameters are defined by the relations

$$
\begin{aligned}
-\varepsilon^i{}_{jk}b^j{}_\mu\theta^k-(\partial_\mu\xi^\rho)b^i{}_\rho-\xi^\rho\partial_\rho b^i{}_\mu &= \delta_0 B^i{}_\mu\,,\\
-(\partial_\mu\theta^i+\varepsilon^i{}_{jk}\omega^j{}_\mu\theta^k)-(\partial_\mu\xi^\rho)\omega^i{}_\rho-\xi^\rho\partial_\rho\omega^i{}_\mu &= \delta_0\Omega^i{}_\mu\,,\\
-\varepsilon^i{}_{jk}\lambda^j{}_\mu\theta^k-(\partial_\mu\xi^\rho)\lambda^i{}_\rho-\xi^\rho\partial_\rho\lambda^i{}_\mu &= \delta_0\Lambda^i{}_\mu\,.
\end{aligned}
$$

By solving these equations, we find the asymptotic parameters for local translations,

$$
\begin{aligned}
&\xi^0=\ell T(\varphi)+\mathcal{O}_2\,, &&\xi^1=-r\partial_2 S(\varphi)+\mathcal{O}_0\,,\\
&\xi^2=S(\varphi)+\mathcal{O}_2\,,
\end{aligned}
\tag{3.3a}
$$

and for local Lorentz rotations:

$$
\begin{aligned}
\theta^0&=-\frac{2\ell}{\sqrt{3(\nu^2+3)(\nu^2-1)}r}\partial_2^2S(\varphi)+\mathcal{O}_2\,,\\
\theta^1&=\frac{2\ell\sqrt{\nu^2+3}}{3(\nu^2-1)r}\partial_2T(\varphi)+\mathcal{O}_3\,,\\
\theta^2&=-\frac{4\ell\nu}{(\nu^2+3)\sqrt{3(\nu^2-1)}}\frac{1}{r}\partial_2^2S(\varphi)+\mathcal{O}_2\,.
\end{aligned}
\tag{3.3b}
$$

These parameters define the symmetry of the (asymptotic) boundary of spacetime, in the spacelike stretched AdS sector of TMG_Λ.

3.3 Asymptotic symmetry

To find the interpretation of the asymptotic parameters, we calculate the commutator algebra of the corresponding gauge transformations. To begin with, we observe that commutator algebra of the local Poincaré transformations (2.2) is closed: $[\delta_0(1),\delta_0(2)]=\delta_0[3]$, where $\delta_0(1):=\delta_0(\xi_1^m,\theta_1^i)$ etc, while the composition rule is given by:

$$
\begin{aligned}
\xi_3^\mu &= \xi_1\cdot\partial\xi_2^\mu-\xi_2\cdot\partial\xi_1^\mu\,,\\
\theta_3^i &= \varepsilon^i{}_{mn}\theta_1^m\theta_2^n+\xi_1\cdot\partial\theta_2^i-\xi_2\cdot\partial\theta_1^i\,.
\end{aligned}
$$

Substituting here the asymptotic parameters (3.3) and comparing the lowest order terms, we obtain:

$$
\begin{aligned}
T_3 &= S_1\partial_2T_2-S_2\partial_2T_1\,,\\
S_3 &= S_1\partial_2S_2-S_2\partial_2S_1\,.
\end{aligned}
\tag{3.4}
$$

To clarify the meaning of this result, it is useful to define the residual or pure gauge transformations as the transformations generated by the higher order terms in (3.3). Pure gauge transformations are known to be irrelevant in the canonical analysis of the asymptotic structure of spacetime [16]. This fact is made more precise by saying that the *asymptotic symmetry group* is defined as the factor group of gauge transformations generated by (3.3), with respect to the residual gauge transformations. In other words, the asymptotic symmetry is defined by the pair (T,S), ignoring all the residual, higher-order terms.

Now, introducing the Fourier expansion of the parameters and the related notation

$$
\begin{aligned}
k_n &:= \delta_0(T = e^{in\varphi}, S = 0)\,, \\
\ell_n &:= \delta_0(T = 0, S = e^{in\varphi})\,,
\end{aligned}
$$

the commutator algebra of the asymptotic transformations takes the form the semi-direct sum of $u(1)_{\rm KM}$ and the Virasoro algebra,

$$
\begin{aligned}
i\,[k_m, k_n] &= 0\,, \\
i\,[k_m, \ell_n] &= m k_{m+n}\,, \\
i\,[\ell_m, \ell_n] &= (m-n)\ell_{m+n}\,.
\end{aligned} \tag{3.5}
$$

The same algebra was also found in [9, 10]. Central charges are here absent, but they will appear in the canonical analysis.

The adopted asymptotic conditions (3.2) are chosen in agreement with the requirements (a) and (b), formulated at the beginning of this section, and the related symmetry structure is encoded in the form of the asymptotic gauge parameters (3.3). The status of the requirement (c) will be clarified in the canonical analysis of the next section.

4 Canonical realization of the asymptotic symmetry

Asymptotic symmetry of a gauge theory is most clearly understood in the framework of the canonical formalism. In this section, we apply the results obtained in [11] to study the canonical aspects of the asymptotic structure of TMG_Λ in the spacelike stretched AdS sector.

Using the Castellani algorithm [17], we found the following expression for the canonical gauge generator [11]:

$$
\begin{aligned}
G &= -G_1 - G_2\,, \\
G_1 &= \dot\xi^\rho\left(b^i{}_\rho\pi_i{}^0 + \lambda^i{}_\rho p_i{}^0 + \omega^i{}_\rho\Pi_i{}^0\right) \\
&\quad + \xi^\rho\left[b^i{}_\rho\bar{\mathcal{H}}_i + \lambda^i{}_\rho\bar{\mathcal{T}}_i + \omega^i{}_\rho\bar{\mathcal{K}}_i + (\partial_\rho b^i_0)\pi_i{}^0 + (\partial_\rho\lambda^i{}_0)p_i{}^0 + (\partial_\rho\omega^i{}_0)\Pi_i{}^0\right]\,, \\
G_2 &= \dot\theta^i\Pi_i{}^0 + \theta^i\left[\bar{\mathcal{K}}_i - \varepsilon_{ijk}\left(b^j{}_0\pi^{k0} + \lambda^j{}_0 p^{k0} + \omega^j{}_0\Pi^{k0}\right)\right]\,.
\end{aligned} \tag{4.1}
$$

Here, the integration symbol $\int d^3x$ is omitted for simplicity, the canonical momenta corresponding to $(b^i{}_\mu, \omega^i{}_\mu, \lambda^i{}_\mu)$, are denoted as $(\pi_i{}^\mu, \Pi_i{}^\mu, p_i{}^\mu)$, and explicit expressions for various terms appearing in G are given in appendix D. The action of the gauge generator G on the fields, defined by $\delta_0\phi = \{\phi, G\}$, has the form (2.2).

4.1 Surface terms

Since canonical generators act on dynamical variables via the Poisson bracket (PB) operation, they must have well-defined functional derivatives. When this is not the case, the problem can be usually solved by adding suitable surface terms [18].

We start by examining the variation of the Lorentz generator G_2:

$$\begin{aligned}\delta G_2 &= \theta^i\delta\mathcal{K}_i + \partial\hat{\mathcal{O}} + R \\ &= -2a\varepsilon^{0\alpha\beta}\partial_\alpha(\theta^i\delta b_{i\beta} + \theta^i\mu^{-1}\delta\omega_{i\beta}) + \partial\hat{\mathcal{O}} + R \\ &= \partial\mathcal{O}_1 + R\,.\end{aligned}$$

Here, $\hat{\mathcal{O}}$ are terms with arbitrarily fast asymptotic decrease, R are regular terms which do not contain variations of the derivatives of fields, and the final result is a consequence of the asymptotic conditions (3.2). Since both $\mathcal{O}_1$ and R terms do not contribute to surface integrals, it follows that G_2 is a well-defined generator.

For G_1, we have:

$$\begin{aligned}\delta G_1 &= \xi^\rho\left(b^i{}_\rho\delta\mathcal{H}_i + \omega^i{}_\rho\delta\mathcal{K}_i + \lambda^i{}_\rho\delta\mathcal{T}_i\right) + \partial\hat{\mathcal{O}} + R \\ &= -\varepsilon^{0\alpha\beta}\partial_\alpha\Big[\xi^\rho b^i{}_\rho\left(2a\delta\omega_{i\beta} + \delta\lambda_{i\beta}\right) \\ &\qquad + \xi^\rho\lambda^i{}_\rho\delta b_{i\beta} + \xi^\rho\omega^i{}_\rho(2a\delta b_{i\beta} + 2a\mu^{-1}\delta\omega_{i\beta})\Big] + \partial\hat{\mathcal{O}} + R\,.\end{aligned}$$

Using the adopted asymptotic conditions, we find:

$$\begin{aligned}\delta G_1 &= -\partial_\alpha(\xi^0\delta\mathcal{E}^\alpha + \xi^2\delta\mathcal{M}^\alpha) + \mathcal{O}_1 + R \\ &= -\delta\partial_\alpha(\xi^0\mathcal{E}^\alpha + \xi^2\mathcal{M}^\alpha) + \mathcal{O}_1 + R\,,\end{aligned}$$

where

$$\begin{aligned}\mathcal{E}^\alpha &= \varepsilon^{0\alpha\beta}\left[b^0{}_0\left(\frac{4a}{3}\omega^0{}_\beta + \lambda^0{}_\beta\right) - b^2{}_0\left(\frac{4a}{3}\omega^2{}_\beta + \lambda^2{}_\beta - \frac{a}{3\ell}\frac{(2\nu^2+3)}{\nu}b^2{}_\beta\right)\right], \\ \mathcal{M}^\alpha &= -\varepsilon^{0\alpha\beta}\left[b^2{}_2(2a\omega^2{}_\beta + \lambda^2{}_\beta) + \frac{a\ell}{3\nu}(\omega^2{}_2\omega^2{}_\beta - \omega^0{}_2\omega^0{}_\beta)\right].\end{aligned} \tag{4.2}$$

Then, after re-introducing the spatial integration, the improved generator takes the form

$$\begin{aligned}\tilde{G} &= G + \Gamma\,, \\ \Gamma &:= -\int_0^{2\pi} d\varphi\left(\ell T\mathcal{E}^1 + S\mathcal{M}^1\right).\end{aligned} \tag{4.3}$$

4.2 Energy and angular momentum

The general relation (4.3) implies:

$$\tilde{G}[\xi^0] = G[\xi^0] - \Gamma[\xi^0]\,, \qquad \tilde{G}[\xi^2] = G[\xi^2] - \Gamma[\xi^2]\,.$$

For $\xi^0 = 1$ and $\xi^2 = 1$, the values of the surface terms have the meaning of energy and angular momentum of the system, respectively:

$$E = \int_0^{2\pi} d\varphi\,\mathcal{E}^1\,, \qquad M = \int_0^{2\pi} d\varphi\,\mathcal{M}^1\,. \tag{4.4}$$

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Let us show that these expressions are *finite*. Using the adopted asymptotic conditions, one can express $\mathcal{E}^1$ and $\mathcal{M}^1$ as functions of the sub-leading terms $(B^i{}_\mu, \Omega^i{}_\mu, \Lambda^i{}_\mu)$:

$$\begin{aligned}
\mathcal{E}^1 &= \frac{4a}{3}\sqrt{\frac{\nu^2+3}{3(\nu^2-1)}}\left(\Omega^0{}_2 + \frac{3}{4a}\Lambda^0{}_2\right) \\
&\quad + \frac{8a\nu}{3\sqrt{3(\nu^2-1)}}\left(\frac{2\nu^2+3}{4\nu}\frac{B^2{}_2}{\ell} - \Omega^2{}_2 - \frac{3}{4a}\Lambda^2{}_2\right) + \mathcal{O}_1\,, \\
\mathcal{M}^1 &= -a\left[B^2{}_2\left(2\Omega^2{}_2 + \frac{1}{a}\Lambda^2{}_2\right) + \frac{\ell}{3\nu}(\Omega^2{}_2 - \Omega^0{}_2)(\Omega^2{}_2 + \Omega^0{}_2)\right] \\
&\quad - \frac{a\sqrt{3(\nu^2-1)}}{2}\left[\frac{B_2^2}{\nu\ell} + \frac{4}{3}\Omega^2{}_2 + \frac{1}{a}\Lambda^2{}_2 + \frac{2\sqrt{\nu^2+3}}{3\nu}\Omega^0{}_2\right] r + \mathcal{O}_1\,.
\end{aligned}$$

Since the sub-leading terms are either constant or tend to zero in the asymptotic region, it follows immediately that $\mathcal{E}^1 = \mathcal{O}_0$, and consequently, the expression for E in (4.4) is finite. In order to prove the finiteness of the angular momentum, we need the improved asymptotic relation (C.3), derived in appendix C. It implies $\mathcal{M}^1 = \mathcal{O}_0$, which completes the proof of finiteness.

Now, we can calculate energy and angular momentum of the spacelike stretched black hole (2.4):

$$\begin{aligned}
E &= \frac{(\nu^2+3)}{24G\ell}\left[r_+ + r_- - \frac{1}{\nu}\sqrt{r_+r_-(3+\nu^2)}\right] \\
M &= -\frac{\nu^2+3}{384G\ell\nu}\left[(\nu^2+3)(r_+ + r_-) + 8(r_+ + r_-)\nu\sqrt{r_+r_-(3+\nu^2)} - 2r_+r_-(11\nu^2+9)\right] \\
&\equiv \frac{\nu(\nu^2+3)}{96G\ell}\left[\left(r_+ + r_- - \frac{1}{\nu}\sqrt{r_+r_-(3+\nu^2)}\right)^2 - \frac{5\nu^2+3}{4\nu^2}(r_+ - r_-)^2\right]\,,
\end{aligned} \tag{4.5}$$

The result coincides with the ADT charges that can be found in [6] (see also [19]).

Returning now to the beginning of this section, where we introduced the concept of the warped AdS asymptotics, we see that our asymptotic conditions (3.2) are also in agreement with the last requirement (c).

4.3 Canonical algebra

Now, we wish to find the PB algebra of the improved canonical generators.

After introducing the notation $\tilde{G}(1) := \tilde{G}[T_1, S_1]$, $\tilde{G}(2) := \tilde{G}[T_2, S_2]$, we use the main theorem of [20] to conclude that the PB $\{\tilde{G}(2), \tilde{G}(1)\}$ of two differentiable generators is also a differentiable generator. This implies

$$\left\{\tilde{G}(2), \tilde{G}(1)\right\} = \tilde{G}(3) + C_{(3)}, \tag{4.6a}$$

where the parameters of $\tilde{G}(3)$ are defined by the composition rule (3.4), while $C_{(3)}$ is an unknown field-independent functional, $C_{(3)} := C_{(3)}[T_1, S_1; T_2, S_2]$, the central term of the canonical algebra. The form of $C_{(3)}$ can be found using the relation

$$\delta_0(1)\Gamma(2) \approx \Gamma(3) + C_{(3)}\,,$$

which is a consequence of $\{\tilde{G}(2),\tilde{G}(1)\}\approx\delta_0(1)\Gamma(2)$. The expression $\delta_0(1)\Gamma(2)$ is calculated using the transformation laws

$$
\begin{aligned}
\delta_0\mathcal{E}^1 &= -S\partial_2\mathcal{E}^1-(\partial_2 S)\mathcal{E}^1-\frac{2a(\nu^2+3)}{3\nu}\partial_2 T\,,\\
\delta_0\mathcal{M}^1 &= -2(\partial_2 S)\mathcal{M}^1-S\partial_2\mathcal{M}^1-(\ell\partial_2 T)\mathcal{E}^1-\frac{2a\ell(5\nu^2+3)}{3\nu(\nu^2+3)}\partial_2^3 S\,.
\end{aligned}
$$

Once we know $\delta_0(1)\Gamma(2)$, we can identify the central term:

$$
C_{(3)}=\frac{2a\ell(\nu^2+3)}{3\nu}\int_0^{2\pi}d\varphi T_2\partial_2 T_1+\frac{2a\ell(5\nu^2+3)}{3\nu(\nu^2+3)}\int_0^{2\pi}d\varphi S_2\partial_2^3 S_1\,. \tag{4.6b}
$$

The form of the canonical algebra (4.6) implies that the improved generator is *conserved*. Indeed, using the relation $\tilde{G}[1,0]=-\ell\tilde{H}_T$ and the composition rule (3.4), we have:

$$
\begin{aligned}
\frac{d}{dt}\tilde{G} &= \frac{\partial}{\partial t}\tilde{G}+\left\{\tilde{G},\tilde{H}_T\right\}\\
&= \frac{\partial}{\partial t}\tilde{G}-\frac{1}{\ell}\left\{\tilde{G}[T,S],\tilde{G}[1,0]\right\}\approx\frac{\partial}{\partial t}\Gamma[T,S]=0\,,
\end{aligned}
$$

since the parameters T and S are time independent. Consequently, we have the conservation of the surface term Γ, and hence, the conservation of the energy and the angular momentum.

After expressing the canonical generator in terms of the Fourier modes,

$$
K_n:=\tilde{G}(T=e^{-in\varphi},S=0)\,,\qquad L_n:=\tilde{G}(T=0,S=e^{-in\varphi})\,,
$$

the canonical algebra (4.6) takes a more familiar form:

$$
\begin{aligned}
i\{K_m,K_n\} &= -\frac{c_K}{12}m\delta_{m,-n}\,,\\
i\{K_m,L_n\} &= mK_{m+n}\,,\\
i\{L_m,L_n\} &= (m-n)L_{m+n}+\frac{c_V}{12}m^3\delta_{m,-n}\,,
\end{aligned} \tag{4.7a}
$$

where

$$
c_K=\frac{(\nu^2+3)\ell}{G\nu}\,,\qquad c_V=\frac{(5\nu^2+3)\ell}{G\nu(\nu^2+3)}\,. \tag{4.7b}
$$

Thus, the canonical realization of the asymptotic symmetry is given as the semi-direct sum of $u(1)_{\mathrm{KM}}$ and the Virasoro algebra, with central charges c_K and c_V.

The authors of [9, 10] found an asymptotic algebra which is essentially the same as ours, up to some minor differences in conventions.

5 Sugawara construction

Clearly, the asymptotic algebra (4.7) does not describe the conformal symmetry, as conjectured in [6]. However, there is a particular construction due to Sugawara [13], which reveals how the conformal algebra can be reconstructed on the basis of (4.7). In this procedure, the presence of central charges is of essential importance.

In the first step, we introduce the set of generators

$$\bar{L}_n := -\frac{6}{c_K}\sum_r K_r K_{n-r}\,, \tag{5.1a}$$

which obey the following PB relations:

$$\begin{aligned} i\left\{K_m, \bar{L}_n\right\} &= mK_{m+n}\,, \\ i\left\{\bar{L}_m, \bar{L}_n\right\} &= (m-n)\bar{L}_{m+n}\,, \\ i\left\{\bar{L}_m, L_n\right\} &= (m-n)\bar{L}_{m+n}\,. \end{aligned}$$

Next, we introduce

$$L_n^- = L_n - \bar{L}_n\,, \tag{5.1b}$$

whereupon (4.7) takes the form of a *direct sum* of $u(1)_{\mathrm{KM}}$ and the Virasoro algebra:

$$\begin{aligned} i\left\{K_m, K_n\right\} &= -\frac{c_K}{12}m\delta_{m,-n}\,, \\ i\left\{K_m, L_n^-\right\} &= 0\,, \\ i\left\{L_m^-, L_n^-\right\} &= (m-n)L_{m+n}^- + \frac{c^-}{12}m^3\delta_{m,-n}\,, \end{aligned}$$

where $c^- := c_V$. Finally, we define

$$-L_n^+ := \bar{L}_{-n} + in\alpha K_{-n}\,, \tag{5.1c}$$

where α is an arbitrary constant. The PB algebra between $L_n^\mp$ takes the well-known form:

$$\begin{aligned} i\left\{L_m^+, L_n^+\right\} &= (m-n)L_{m+n}^+ + \frac{c^+}{12}m^3\delta_{m,-n}\,, \\ i\left\{L_m^+, L_n^-\right\} &= 0\,, \\ i\left\{L_m^-, L_n^-\right\} &= (m-n)L_{m+n}^- + \frac{c^-}{12}m^3\delta_{m,-n}\,, \end{aligned} \tag{5.2}$$

where $c^+ := c_K\alpha^2$. This result reveals the conformal structure hidden in (4.7).

Clearly, the value of α in c^+ has to be fixed by some additional requirements. Before going to that, we display here the values of $L_0^\pm$ in terms of the canonical energy and angular momentum for the spacelike warped black hole:

$$\begin{aligned} L_0^+ &= \tfrac{6G\nu\ell}{\nu^2+3}E^2 &&= \frac{(\nu^2+3)\nu}{96G\ell}\left[r_+ + r_- - \frac{1}{\nu}\sqrt{r_+r_-(3+\nu^2)}\right]^2\,, \\ L_0^- &= L_0^+ - M &&= \frac{(\nu^2+3)(5\nu^2+3)}{384\nu G\ell}(r_+ - r_-)^2\,. \end{aligned} \tag{5.3}$$

These results are in complete agreement with those found in [6].

In order to find out the value of α, one can use our central charges $c^\mp$ to calculate the black hole entropy via Cardy's formula:

$$S_{\mathrm{c}} = 2\pi\sqrt{\frac{L_0^+c^+}{6}} + 2\pi\sqrt{\frac{L_0^-c^-}{6}}\,.$$

A direct calculation leads to

$$S_{\rm c} = 2\pi\alpha\frac{(\nu^2+3)}{24G}\left[r_+ + r_- - \frac{1}{\nu}\sqrt{r_+r_-(3+\nu^2)}\right] + \frac{\pi(5\nu^2+3)}{24\nu G}(r_+ - r_-)\,. \tag{5.4}$$

On the other hand, the gravitational black hole entropy of TMG_Λ has the form [6, 15]:

$$S_{\rm gr} = \frac{\pi}{24\nu G}\left[(9\nu^2+3)r_+ - (\nu^2+3)r_- - 4\nu\sqrt{r_+r_-(\nu^2+3)}\right]\,. \tag{5.5}$$

Comparing $S_{\rm c}$ and $S_{\rm gr}$, one finds that $S_{\rm c} = S_{\rm gr}$ for

$$\alpha = \frac{2\nu}{\nu^2+3} \equiv \frac{2\ell}{Gc_K}\,. \tag{5.6}$$

Consequently, the values of the central charges in the Virasoro algebras (5.2) are the same as those conjectured in [6]:

$$c^- = \frac{(5\nu^2+3)\ell}{G\nu(\nu^2+3)}\,, \qquad c^+ = \frac{4\nu\ell}{G(\nu^2+3)}\,. \tag{5.7}$$

In conclusion, our main result is expressed by the formulas (5.2) and (5.7), and it confirms the hypothesis formulated heuristically in [6], at least at the classical level.

6 Concluding remarks

In this paper, we analyzed asymptotic structure of TMG_Λ in the spacelike stretched AdS sector.

(1) We introduced spacelike stretched AdS asymptotic conditions and found the form of the corresponding asymptotic parameters. The commutator algebra of the asymptotic transformations is the semi-direct sum of $u(1)_{\rm KM}$ with the Virasoro algebra, without central charges, which is a natural generalization of the vacuum isometry algebra $u(1) \oplus sl(2,R)$. Asymptotic conditions for the metric recently proposed in [10] coincide with ours.

(2) With the adopted asymptotic conditions, we constructed the improved canonical generators and found the expressions for the conserved charges. In particular, we calculated the energy and angular momentum of the spacelike stretched black hole. We showed that canonical algebra of the improved generators takes the form of the semi-direct product of $u(1)_{\rm KM}$ and the Virasoro algebra, with two central charges. Our algebra has essentially the same form as the one found in [9, 10] by different methods.

(3) In the last step, we used the Sugawara construction in the $u(1)_{\rm KM}$ sector to show that the asymptotic dynamics of TMG_Λ can be described by the conformal symmetry, realized by two independent Virasoro algebras with different central charges. This result proves that the hypothesis formulated in [6] is correct, at least at the classical level.

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A Spacelike stretched AdS_3

Here, we describe some basic properties of spacelike warped AdS$_3$ solutions [5, 6].

Maximally symmetric solution of GR$_\Lambda$ for $\Lambda < 0$, the anti-de Sitter space AdS$_3$, is also a solution of TMG$_\Lambda$. It can be represented as a hypersphere embedded in a four-dimensional flat space M_4 with the metric $\eta = (+,+,-,-)$. By construction, the isometry group of AdS$_3$ is SO$(2,2) \sim$ SL$(2,R) \times$ SL$(2,R)$, and we denote the corresponding Killing vectors by $(\xi_{(0)}, \xi_{(1)}, \xi_{(2)})$ and $(\bar\xi_{(0)}, \bar\xi_{(1)}, \bar\xi_{(2)})$, respectively. After introducing a convenient set of coordinates (τ, u, σ), analogous to the Euler angles for the 3-sphere, the metric of AdS$_3$ can be written in the form

$$ds^2 = \frac{\ell^2}{4}\left[\cosh\sigma^2 d\tau^2 - d\sigma^2 - (du + \sinh\sigma d\tau)^2\right] ,$$

where $\{\tau, u, \sigma\}$ are in the range $(-\infty, +\infty)$.

The metric of the spacelike warped AdS$_3$ is given by:

$$ds^2 = \frac{\ell^2}{\nu^2+3}\left[\cosh\sigma^2 d\tau^2 - d\sigma^2 - \frac{4\nu^2}{\nu^2+3}(du + \sinh\sigma d\tau)^2\right] , \tag{A.1}$$

where $w := 4\nu^2/(\nu^2+3)$ is the warp factor. The isometry group of (A.1) is generated by four Killing vectors,

$$\begin{aligned}
\xi_{(2)} &= 2\partial_u ,\\
\bar\xi_{(1)} &= 2\sin\tau\tanh\sigma\partial_\tau - 2\cos\tau\partial_\sigma + \frac{2\sin\tau}{\cosh\sigma}\partial_u ,\\
\bar\xi_{(2)} &= -2\cos\tau\tanh\sigma\partial_\tau - 2\sin\tau\partial_\sigma - \frac{2\cos\tau}{\cosh\sigma}\partial_u ,\\
\bar\xi_{(0)} &= 2\partial_\tau ,
\end{aligned} \tag{A.2}$$

which satisfy the commutator algebra $u(1) \times sl(2,R)$. For $\nu^2 > 1$, we have $w > 1$, and the metric (A.1) describes the *spacelike stretched* AdS$_3$.

One can show that the spacelike stretched AdS$_3$ is locally isometric to the black hole (2.4), by using the following change of coordinates:

$$\begin{aligned}
\tau &= \arctan\left[\frac{2\sqrt{(r-r_+)(r-r_-)}\sinh\phi}{2r - r_+ - r_-}\right] ,\\
\ell u &= \frac{\nu^2+3}{4\nu}\left[2t + \ell\left(\nu(r_+ + r_-) - \sqrt{r_+ r_-(\nu^2+3)}\right)\varphi\right] - \ell \operatorname{atanh}\left[\frac{r_+ + r_- - 2r}{r_+ - r_-}\coth\phi\right]\\
\sigma &= \operatorname{asinh}\left[\frac{2\sqrt{(r-r_+)(r-r_-)}\cosh\phi}{r_+ - r_-}\right] ,
\end{aligned} \tag{A.3}$$

where

$$\phi = \frac{(3+\nu^2)(r_+ - r_-)}{4\ell}\varphi\,,$$

and (t, r, φ) are the usual Schwarzschild-like coordinates. Note that the black hole (2.4) is obtained from the spacelike stretched AdS_3 by the identification $\varphi \sim \varphi + 2\pi$. Expressed in terms of the new coordinates, the Killing vectors (A.2) take the form:

$$\begin{aligned}
\xi_{(2)} &= \frac{4\nu^2}{\nu^2+3}\ell\partial_t\,,\\
\bar\xi_{(1)} &= -2\frac{\sqrt{r_+r_-(\nu^2+3)}(r_++r_--2r)+2\nu\left[r(r_++r_-)-2r_+r_-\right]}{(3+\nu^2)(r_+-r_-)\sqrt{(r-r_+)(r-r_-)}}\sinh\phi\ell\partial_t\,,\\
&\quad -2\sqrt{(r-r_+)(r-r_-)}\cosh\phi\partial_r+\frac{4\ell(2r-r_+-r_-)}{(3+\nu^2)(r_+-r_-)\sqrt{(r-r_+)(r-r_-)}}\sinh\phi\partial_\varphi\,,\\
\bar\xi_{(2)} &= -4\frac{\sqrt{r_+r_-(\nu^2+3)}-\nu(r_++r_-)}{(3+\nu^2)(r_+-r_-)}\ell\partial_t-\frac{8\ell}{(3+\nu^2)(r_+-r_-)}\partial_\varphi\,,\\
\bar\xi_{(0)} &= -2\frac{\sqrt{r_+r_-(\nu^2+3)}(r_++r_--2r)+2\nu\left[r(r_++r_-)-2r_+r_-\right]}{(3+\nu^2)(r_+-r_-)\sqrt{(r-r_+)(r-r_-)}}\cosh\phi\ell\partial_t\,,\\
&\quad -2\sqrt{(r-r_+)(r-r_-)}\sinh\phi\partial_r-\frac{4\ell(r_++r_--2r)}{(3+\nu^2)(r_+-r_-)\sqrt{(r-r_+)(r-r_-)}}\cosh\phi\partial_\varphi\,.
\end{aligned} \tag{A.4}$$

After the identification $\varphi \sim \varphi + 2\pi$, only $\xi_{(2)}$ and $\bar\xi_{(2)}$ remain the Killing vectors of the black hole (2.4).

The asymptotic form of $\xi_{(2)}$ and $\bar\xi_{(2)}$ is quite simple, while for $\bar\xi_{(1)}$ and $\bar\xi_{(0)}$ we have:

$$\begin{aligned}
\bar\xi_{(1)} &= -4\left[\frac{\nu(r_++r_-)-\sqrt{(\nu^2+3)r_+r_-}}{(3+\nu^2)(r_+-r_-)}\sinh\phi+\mathcal{O}_2\right]\ell\partial_t\\
&\quad -(2\cosh\phi r+\mathcal{O}_0)\partial_r+\left[\frac{8\ell}{(3+\nu^2)(r_+-r_-)}\sinh\phi+\mathcal{O}_2\right]\partial_\varphi\,,\\
\bar\xi_{(0)} &= -4\left[\frac{\nu(r_++r_-)-\sqrt{(\nu^2+3)r_+r_-}}{(3+\nu^2)(r_+-r_-)}\cosh\phi+\mathcal{O}_1\right]\ell\partial_t\,,\\
&\quad -(2r\sinh\phi+\mathcal{O}_0)\partial_r+\left[\frac{8\ell\cosh\phi}{(3+\nu^2)(r_+-r_-)}+\mathcal{O}_2\right]\partial_\varphi\,.
\end{aligned}$$

These expressions are needed for our discussion of the asymptotic conditions in section 3.

B The curvature, Ricci and Cotton tensors

In this appendix, we present some technical details related to the form of the spacelike warped black hole solution in the first order formalism.

Using the connection (2.5b), we find that the curvature R_i is given by

$$\begin{aligned}
R_0 &= \left(\beta'B+2\beta\gamma\right)b^0b^1-\left(\gamma'B+\gamma^2+\beta^2\right)b^1b^2\,,\\
R_1 &= -(\alpha\gamma+\beta^2)b^2b^0\,,\\
R_2 &= -\left(\alpha'B+\alpha^2-3\beta^2\right)b^0b^1-\left(\beta'B+2\beta\gamma\right)b^1b^2\,,
\end{aligned} \tag{B.1a}$$

or equivalently:

$$
\begin{aligned}
R_0 &= -\frac{3}{\ell^2}(\nu^2-1)NKN_\varphi b^0 b^1 - \frac{1}{\ell^2}\left(\nu^2+3(\nu^2-1)N^2\right) b^1 b^2\,, \\
R_1 &= -\frac{\nu^2}{\ell^2} b^2 b^0\,, \\
R_2 &= -\frac{1}{\ell^2}\left(3-2\nu^2-3(\nu^2-1)N^2\right) b^0 b^1 + \frac{3}{\ell^2}(\nu^2-1)NKN_\varphi b^1 b^2\,,
\end{aligned} \tag{B.1b}
$$

Then, the components of the Ricci tensor $(Ric)_{mn} = -\varepsilon^{kl}{}_m R_{kln}$ are found to be:

$$
\begin{aligned}
&(Ric)_{00} = \alpha' B + \alpha^2 + \alpha\gamma - 2\beta^2\,, && (Ric)_{01} = 0\,, \\
&(Ric)_{11} = -\alpha' B - \gamma' B - \alpha^2 - \gamma^2 + 2\beta^2\,, && (Ric)_{12} = 0\,, \\
&(Ric)_{22} = -\gamma' B - \gamma^2 - \alpha\gamma - 2\beta^2\,, && (Ric)_{20} = -(\beta' B + 2\beta\gamma)\,,
\end{aligned} \tag{B.2a}
$$

or equivalently:

$$
\begin{aligned}
&(Ric)_{00} = \frac{1}{\ell^2}\left(3-\nu^2-3(\nu^2-1)N^2\right)\,, && (Ric)_{01} = 0\,, \\
&(Ric)_{11} = -\frac{1}{\ell^2}(3-\nu^2)\,, && (Ric)_{12} = 0\,, \\
&(Ric)_{22} = -\frac{1}{\ell^2}\left(2\nu^2+3(\nu^2-1)N^2\right)\,, && (Ric)_{20} = \frac{3}{\ell^2}(\nu^2-1)NKN_\varphi\,.
\end{aligned} \tag{B.2b}
$$

Finally, the Cotton 2-form reads:

$$
\begin{aligned}
C_0 &= \frac{9\nu}{\ell^3}(\nu^2-1)NKN_\varphi b^0 b^1 - \frac{3\nu}{\ell^3}(\nu^2-1)(3N^2-1) b^1 b^2\,, \\
C_1 &= \frac{3\nu}{\ell^3}(\nu^2-1) b^2 b^0\,, \\
C_2 &= -\frac{3\nu}{\ell^3}(\nu^2-1)(2+3N^2) b^0 b^1 - \frac{9\nu}{\ell^3}(\nu^2-1)NKN_\varphi b^1 b^2\,.
\end{aligned} \tag{B.3}
$$

C Improved asymptotic conditions

Our asymptotic conditions (3.2) are chosen so that all higher-order terms are left completely arbitrary. However, this feature can be improved by noting that expressions that vanish on shell should have an arbitrarily fast asymptotic decrease. By applying this principle to the secondary constraints, we obtain the following relations between higher-order terms:

$$
\begin{aligned}
&\partial_0 \Omega^0{}_2 = \mathcal{O}_1\,, && \partial_0 \Lambda^0{}_2 = \mathcal{O}_1\,, && \\
&\partial_0 B^2{}_2 = \mathcal{O}_1\,, && \partial_0 \Omega^2{}_2 = \mathcal{O}_1\,, && \partial_0 \Lambda^2{}_2 = \mathcal{O}_1\,,
\end{aligned} \tag{C.1}
$$

and also

$$
\frac{\nu}{\ell}B^2{}_2+\Omega^2{}_2+\frac{\sqrt{3(\nu^2-1)(\nu^2+3)}}{2\ell}\left(-\frac{\nu}{\ell}B^1{}_1+\Omega^1{}_1\right)r^2=\mathcal{O}_1\,, \tag{C.2a}
$$

$$
\begin{aligned}
&-\frac{\nu}{\sqrt{\nu^2+3}}\left(\Omega^0{}_2+\frac{3}{2a}\Lambda^0{}_2\right)\\
&+\frac{\sqrt{3(\nu^2+3)(\nu^2-1)}}{2\ell}\left(-\frac{2\nu}{\ell}B^1{}_1+\Omega^1{}_1\right)r^2=\mathcal{O}_1\,,
\end{aligned} \tag{C.2b}
$$

$$
\begin{aligned}
&\frac{3-2\nu^2}{2\ell}B^2{}_2+\nu\Omega^2{}_2+\frac{3\nu}{2a}\Lambda^2{}_2\\
&+\frac{\sqrt{3(\nu^2-1)(\nu^2+3)}}{2\ell}\left(\frac{3(2\nu^2+1)}{2\ell}B^1{}_1-\nu\Omega^1{}_1+\frac{3\nu}{2a}\Lambda^1{}_1\right)r^2=\mathcal{O}_1\,,
\end{aligned} \tag{C.2c}
$$

$$
\begin{aligned}
&\frac{1}{\ell}B^2{}_2+\frac{(4\nu^2+3)}{6\nu}\Omega^2{}_2+\frac{\nu}{2a}\Lambda^2{}_2\\
&+\frac{\sqrt{3(\nu^2-1)(\nu^2+3)}}{2\ell}\left(\frac{1}{\ell}B^1{}_1+\frac{1}{2\nu}\Omega^1{}_1-\frac{\nu}{2a}\Lambda^1{}_1\right)r^2=\mathcal{O}_1\,,
\end{aligned} \tag{C.2d}
$$

$$
\begin{aligned}
&-\frac{\nu}{\sqrt{\nu^2+3}}\left(\frac{4\nu^2+3}{6\nu^2}\Omega^0{}_2+\frac{1}{2a}\Lambda^0{}_2\right)\\
&+\frac{\sqrt{3(\nu^2+3)(\nu^2-1)}}{2\ell}\left(\frac{\Omega^1{}_1}{3}+\frac{\Lambda^1{}_1}{2a}\right)r^2=\mathcal{O}_1\,.
\end{aligned} \tag{C.2e}
$$

From (C.2b) and (C.2e), we obtain

$$
-\frac{\nu}{\sqrt{\nu^2+3}}\frac{2\nu^2+3}{2\nu^2}\Omega^0{}_2+\frac{\sqrt{3(\nu^2+3)(\nu^2-1)}}{2\ell}\left(\frac{2\nu}{\ell}B^1{}_1+\frac{3}{2a}\Lambda^1{}_1\right)r^2=\mathcal{O}_1\,. \tag{C.2f}
$$

By eliminating $B^1{}_1$, $\Omega^1{}_1$ and $\Lambda^1{}_1$ from the remaining equations in (C.2), one finds the relation

$$
\frac{\nu}{\ell}B^2{}_2+\frac{4}{3}\Omega^2{}_2+\frac{1}{a}\Lambda^2{}_2+\frac{2\sqrt{\nu^2+3}}{3\nu}\Omega^0{}_2=\mathcal{O}_1\,, \tag{C.3}
$$

that ensures finiteness of the angular momentum.

D Hamiltonian and constraints

In this appendix, we present a brief overview of the canonical structure of TMG_Λ [11].

Starting with the Lagrangian variables $(b^i{}_\mu,\omega^i{}_\mu,\lambda^i{}_\mu)$ and the corresponding canonical momenta $(\pi_i{}^\mu,\Pi_i{}^\mu,p_i{}^\mu)$, we find the following primary constraints:

$$
\begin{aligned}
\phi_i{}^0 &:= \pi_i{}^0\approx 0\,, &\qquad \phi_i{}^\alpha &:= \pi_i{}^\alpha-\varepsilon^{0\alpha\beta}\lambda_{i\beta}\approx 0\,,\\
\Phi_i{}^0 &:= \Pi_i{}^0\approx 0\,, & \Phi_i{}^\alpha &:= \Pi_i{}^\alpha-a\varepsilon^{0\alpha\beta}(2b_{i\beta}+\mu^{-1}\omega_{i\beta})\approx 0\,.\\
p_i{}^\mu &\approx 0\,.
\end{aligned} \tag{D.1}
$$

The canonical Hamiltonian has the form (up to a 3-divergence):

$$\begin{aligned}
\mathcal{H}_c &= b^i{}_0\mathcal{H}_i + \omega^i{}_0\mathcal{K}_i + \lambda^i{}_0\mathcal{T}^i\,,\\
\mathcal{H}_i &= -\varepsilon^{0\alpha\beta}\left(aR_{i\alpha\beta} - \Lambda\varepsilon_{ijk}b^j{}_\alpha b^k{}_\beta + \nabla_\alpha\lambda_{i\beta}\right)\,,\\
\mathcal{K}_i &= -\varepsilon^{0\alpha\beta}\left(aT_{i\alpha\beta} + a\mu^{-1}R_{i\alpha\beta} + \varepsilon_{ijk}b^j{}_\alpha\lambda^k{}_\beta\right)\,,\\
\mathcal{T}_i &= -\frac{1}{2}\varepsilon^{0\alpha\beta}T_{i\alpha\beta}\,.
\end{aligned}$$

After constructing the total Hamiltonian $\mathcal{H}_T$, the consistency requirements on the primary constraints produce the secondary constraints,

$$\mathcal{H}_i \approx 0\,,\qquad \mathcal{K}_i \approx 0\,,\qquad \mathcal{T}_i \approx 0\,, \tag{D.2}$$

and yield the additional relations which determine the multipliers $u^i{}_\alpha, v^i{}_\alpha$ and $w^i{}_\alpha$. The modified total Hamiltonian takes the form (up to a 3-divergence):

$$\begin{aligned}
\mathcal{H}'_T &= b^i{}_0\bar{\mathcal{H}}_i + \omega^i{}_0\bar{\mathcal{K}}_i + \lambda^i{}_0\bar{\mathcal{T}}_i + u^i{}_0\pi_i{}^0 + v^i{}_0\Pi_i{}^0 + w^i{}_0 p_i{}^0\,,\\
\bar{\mathcal{H}}_i &= \mathcal{H}_i - \nabla_\beta\phi_i{}^\beta - \frac{\mu}{2a}\varepsilon_{ijk}\lambda^j{}_\beta\Phi^{k\beta} + \varepsilon_{ijk}\left(2\Lambda b^j{}_\beta + \mu\lambda^j{}_\beta\right)p^{k\beta}\,,\\
\bar{\mathcal{K}}_i &= \mathcal{K}_i - \varepsilon_{ijk}b^j{}_\beta\phi^{k\beta} - \nabla_\beta\Phi_i{}^\beta - \varepsilon_{ijk}\lambda^j{}_\beta p^{k\beta}\,,\\
\bar{\mathcal{T}}_i &= \mathcal{T}_i - \frac{\mu}{2a}\varepsilon_{ijk}b^j{}_\beta\Phi^{k\beta} - \nabla_\beta p_i{}^\beta + \mu\varepsilon_{ijk}b^j{}_\beta p^{k\beta}\,.
\end{aligned}$$

The consistency conditions of the secondary constraints lead to three independent tertiary constraints,

$$\theta_{0\beta} := \lambda_{0\beta} - \lambda_{\beta 0} \approx 0\,, \tag{D.3a}$$

$$\theta_{\alpha\beta} := \lambda_{\alpha\beta} - \lambda_{\beta\alpha} \approx 0\,, \tag{D.3b}$$

while the consistency of $\theta_{\alpha\beta}$ yields a new, quartic constraint:

$$\Psi = 3\Lambda + \mu\lambda \approx 0\,. \tag{D.4}$$

Further consistency requirements determine the multipliers $w^i{}_0{}' := w^i{}_0 - u^k{}_0\lambda_k{}^i$, whereby the consistency procedure is completed. The final form of the total Hamiltonian reads:

$$\begin{aligned}
\hat{\mathcal{H}}_T &= \bar{\mathcal{H}}_T + u^i{}_0\pi_i{}^{0\prime} + v^i{}_0\Pi_i{}^0\,, \qquad\qquad (D.5)\\
\bar{\mathcal{H}}_T &:= b^i{}_0\bar{\mathcal{H}}_i + \omega^i{}_0\bar{\mathcal{K}}_i + \lambda^i{}_0\bar{\mathcal{T}}_i + \bar{w}'_{\beta 0}p^{\beta 0} + \bar{w}'_{00}p^{00}\,.
\end{aligned}$$

where $\pi_i{}^{0\prime} := \pi_i{}^0 + \lambda_i{}^k p_k{}^0$, and the multipliers with an overbar are determined.

As far as the classification of constraints is concerned, we find that $\pi_i{}^{0\prime}, \Pi_i{}^0$ and

$$\begin{aligned}
\hat{\mathcal{H}}_i &= \bar{\mathcal{H}}_i + \lambda_i{}^k\bar{\mathcal{T}}_k + h_i{}^\rho(\nabla_\rho\lambda_{jk})b^k{}_0 p^{j0}\,,\\
\hat{\mathcal{K}}_i &= \bar{\mathcal{K}}_i - \varepsilon_{ijk}(\lambda^j{}_0 p^{k0} - b^j{}_0\lambda^k{}_n p^{n0})\,,
\end{aligned} \tag{D.6}$$

are first class, while all the others are second class.

The canonical generator of gauge transformations has the form (4.1).

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